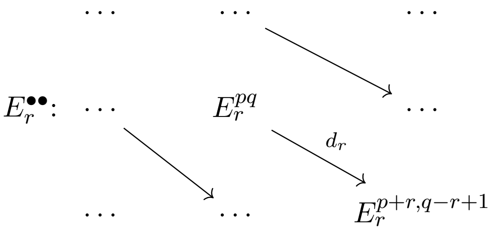
Notation

Throughout:

- We work over a field k with vector spaces.
- Spectral sequences will be cohomologically indexed.



(arrows on page r go r right and $r-1$ down)

- Homology and cohomology are indicated with lowercase h.

Definitions

- A k-vector space H^* is **graded** if $H^* \cong \bigoplus_{n \in \mathbf{N}} H^n$ where H^n is a vector space.
- A graded vector space H^* is a **graded algebra** if there is a product $\varphi : H^p \otimes_k H^q \to H^{p+q}$ for all p and q. The product must be associative. We write $\varphi(a,b)$ as $a \cdot b$.
- Algebras will often have a unit.
- A vector space $E^{\bullet\bullet}$ is **bigraded** if $E^{\bullet\bullet} \cong \bigoplus_{(p,q) \in \mathbf{N} \times \mathbf{N}} E^{pq}$ where E^{pq} is a vector space.
- A bigraded vector space $E^{\bullet\bullet}$ is a **bigraded algebra** if there is an associative product $\varphi : E^{mn} \otimes_k E^{rs} \to E^{m+r,n+s}$ for all m, n, r, s. Write $\varphi(a,b) = a \cdot b$.

Example

For an example of a bigraded algebra, take (A^*, φ) and (B^*, ψ) to be graded algebras with products. Let $E^{pq} = A^p \otimes B^q$ and observe that the following gives a product on E^{pq}:

$$\begin{aligned} E^{pq} \otimes E^{rs} &= A^p \otimes B^q \otimes A^r \otimes B^s \\ &\xrightarrow{\mathrm{id} \otimes T(\cdot \otimes \cdot) \otimes \mathrm{id}} A^p \otimes A^r \otimes B^q \otimes B^s \\ &\xrightarrow{\varphi \otimes \psi} A^{p+r} \otimes B^{q+s} \\ &= E^{p+r, q+s}, \end{aligned}$$

where $T(b \otimes a) = (-1)^{\deg a \deg b} a \otimes b$.

Definitions

- A graded algebra $(H^*, \cdot)$ is a **differential graded algebra** if there exists a degree 1 linear map $d : H^* \to H^*$ such that

$$d(a \cdot b) = d(a) \cdot b + (-1)^{\deg a} a \cdot d(b).$$

- Such a d is called a **derivation**.
- A bigraded algebra $(E^{\bullet\bullet}, \cdot)$ is a **differential bigraded algebra** if there exists a derivation

$$d : \bigoplus_{p+q=n} E^{pq} \to \bigoplus_{r+s=n+1} E^{rs}$$

such that

$$d(a \cdot b) = d(a) \cdot b + (-1)^{p+q} a \cdot d(b)$$

for all $a \in E^{pq}$ and $b \in E^{p'q'}$.

Example

For an example of a differential bigraded algebra, take two differential graded algebras $(A^*, \cdot, d_A)$ and $(B^*, \cdot, d_B)$. Let

$$E^{\bullet\bullet} = A^* \otimes B^*$$

and let the differential d_E on $E^{\bullet\bullet}$ be

$$d_E(\alpha \otimes \beta) = d_A(\alpha) \otimes \beta + (-1)^{\deg \alpha} \alpha \otimes d_B(\beta).$$

Then one can check that

$$d_E(\alpha \otimes \beta \cdot \gamma \otimes \delta) = d_E(\alpha \otimes \beta) \cdot \gamma \otimes \delta + (-1)^{p+q} \alpha \otimes \beta \cdot d_E(\gamma \otimes \delta)$$

for $\alpha \otimes \beta \in E^{pq}$, $\gamma \otimes \delta \in E^{p'q'}$.

Definitions

- A spectral sequence $\{E_r^{\bullet\bullet}, d_r\}$ is a **spectral sequence of algebras** if for each r, $(E_r^{\bullet\bullet}, \varphi_r, d_r)$ is a differential bigraded algebra and the product φ_{r+1} on $E_{r+1}^{\bullet\bullet}$ is induced by the product φ_r of $E_r^{\bullet\bullet}$ on homology. In other words, φ_{r+1} is the composition

$$E_{r+1}^{\bullet\bullet} \otimes E_{r+1}^{\bullet\bullet} = h(E_r^{\bullet\bullet}) \otimes h(E_r^{\bullet\bullet}) \cong h(E_r^{\bullet\bullet} \otimes E_r^{\bullet\bullet}) \xrightarrow{h(\varphi_r)} h(E_r^{\bullet\bullet}) = E_{r+1}^{\bullet\bullet}.$$

- Given a filtration Fil^* of a graded algebra (H^*, φ), the filtration is **stable** with respect to φ if

$$\varphi(Fil^r H^* \otimes Fil^s H^*) \subseteq Fil^{r+s} H^*.$$

- A spectral sequence of algebras $\{E_r^{\bullet\bullet}, d_r\}$ **converges to H^* as a graded algebra** if there is a stable filtration on H^* for which $E_\infty^{\bullet\bullet}$ is isomorphic as a bigraded algebra to the associated bigraded algebra

$$Gr^p(H^*) = {}^{Fil^p H^*}\!\big/_{Fil^{p+1} H^*}.$$

Example

Suppose $E_2^{\bullet\bullet}$ is given as an algebra by

$$E_2^{\bullet\bullet} \cong \mathbf{Q}[x, y, z] \big/ (x^2 = y^4 = z^2 = 0)$$

where $\deg x = (7, 1)$, $\deg y = (3, 0)$, $\deg z = (0, 2)$, $d_2(x) = y^3$, and $d_3(z) = y$.

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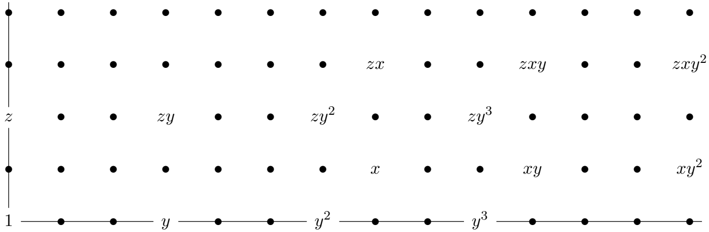
We will see that the spectral sequence collapses at E_4 and xy survives to E_∞ even though x and y do not.

Example

First we draw page 2 (focusing only on generators of the algebra)



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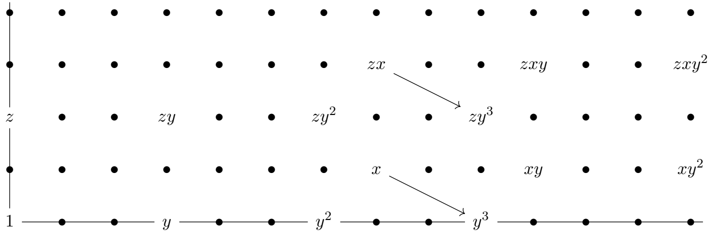

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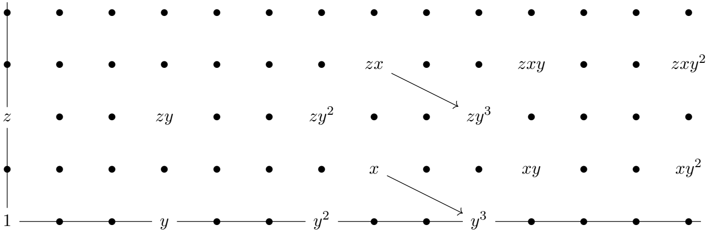
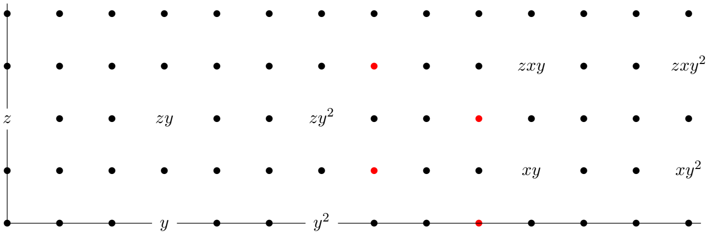


Since d_2 above takes basis elements to basis elements, it is an isomorphism of vector spaces. Thus homology vanishes there, and page E_3 is as follows:

Example

Here is page 3:



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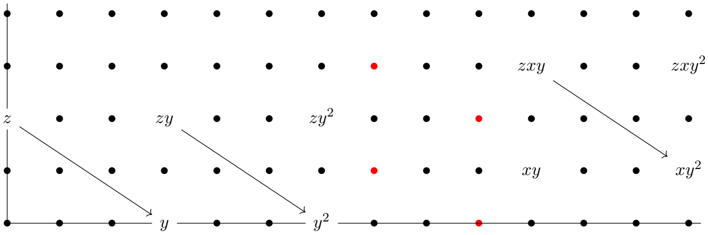

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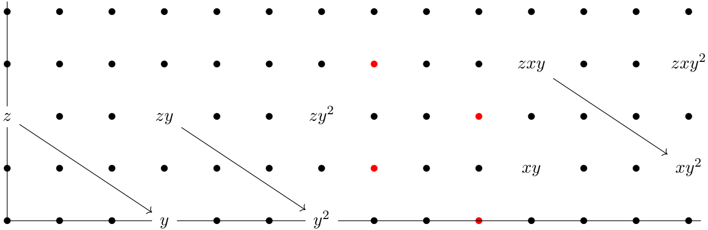
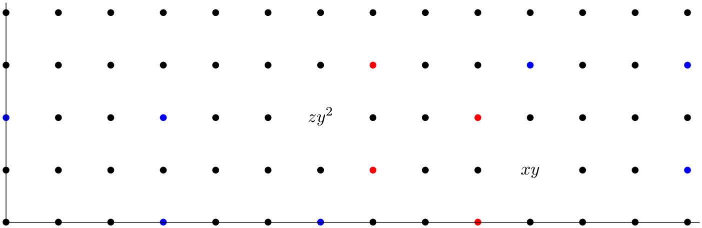


Once again, d_3 takes basis elements to basis elements, hence is an isomorphism. We can now draw page E_4:

Example

Here is page 4:



Example

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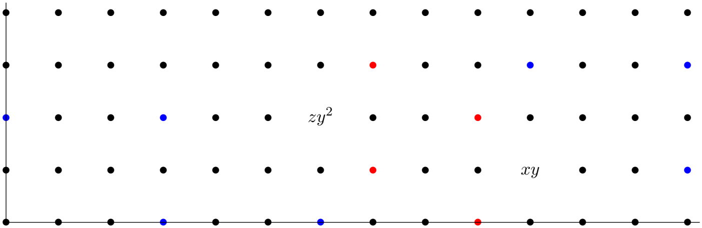

There are no nonzero differentials at this stage. Thus the spectral sequence has abutted at page 4 to the infinite page. We can observe here that indeed, xy survived, but x and y did not.

Definitions

- If (Γ^*, φ) is a graded algebra, then a graded vector space H^* is a Γ^***-module** if the module scalar multiplication

$$m : \Gamma^* \otimes H^* \to H^*$$

respects the multiplication of Γ^*; i.e.,

$$\begin{array}{ccc} \Gamma^* \otimes \Gamma^* \otimes H^* & \xrightarrow{\mathrm{id} \otimes m} & \Gamma^* \otimes H^* \\ \varphi\otimes\mathrm{id}\downarrow & & \downarrow m \\ \Gamma^* \otimes H^* & \xrightarrow[m]{} & H^* \end{array}$$

commutes.

- If Γ^* is a graded algebra and $E^{\bullet\bullet}$ is a bigraded vector space, then Γ^* **acts vertically on** $E^{\bullet\bullet}$ if for all $n \geq 0$, $E^{n\bullet}$ is a Γ^*-module. In other words, there is a scalar multiplication map

$$m_n : \Gamma^* \otimes E^{n\bullet} \to E^{n\bullet}$$

for each n. It is called vertical since

$$m_n : \Gamma^s \otimes E^{nt} \to E^{n,s+t}.$$

Example

For an example of a Γ^* acting vertically on $E^{\bullet\bullet}$, consider a filtered graded vector space/Γ^*-module, call it H^*. Suppose further that the Γ^*-action is *filtration preserving*; i.e.,

$$\Gamma^* \otimes Fil^p H^* \to Fil^p H^*.$$

One can check that Γ^* acts vertically on the associated graded vector space

$$Gr^p(H^*) = {}^{Fil^p H^*}\!\big/_{Fil^{p+1} H^*}.$$

Definitions

- A graded algebra Γ^* **acts on a spectral sequence** $\{E_r^{\bullet\bullet}, d_r\}$ if
  (1) Γ^* acts on $E_r^{\bullet\bullet}$ for each r,
  (2) d_r is Γ^*-linear for each r, and
  (3) the Γ^*-action on $E_{r+1}^{\bullet\bullet}$ is induced through homology from the action of Γ^* on $E_r^{\bullet\bullet}$.
- A spectral sequence $\{E_r^{\bullet\bullet}, d_r\}$ **converges to H^* as a Γ^*-module** if
  (1) $E_r^{\bullet\bullet} \Rightarrow H^*$,
  (2) Γ^* acts on H^*, and
  (3) the filtration Fil^* on H^* induces a Γ^*-action on $Gr^p(H^*) = {Fil^p H^*}/{Fil^{p+1} H^*}$ that is isomorphic to the Γ^*-action on $E_\infty^{\bullet\bullet}$.

Example

Suppose $\Gamma^* = \mathbf{Q}[a, b]$ with $\deg a = 2$, $\deg b = 5$. Suppose there is a spectral sequence with $E_2^{\bullet\bullet}$ such that

- $E_2^{\bullet\bullet}$ is a Γ^*-module,
- its Γ^*-module generators are $\{x, y, z, w\}$ with $\deg x = (8, 4)$, $\deg y = (6, 0)$, $\deg z = (0, 4)$, $\deg w = (10, 1)$, and
- except for that $bx = 0$, Γ^* acts freely on this basis.

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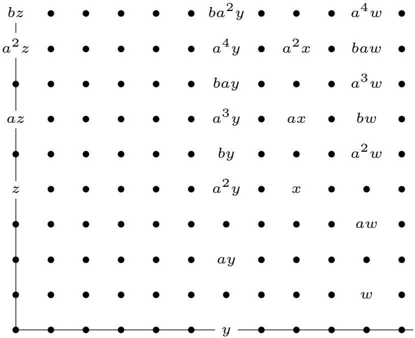
- except for that $bx = 0$, Γ^* acts freely on this basis.

We will see that the spectral sequence collapses at E_2.

Example

Before drawing page 2, note that it is enough to show that $d_r = 0$ on basis elements for all $r > 2$, since d_r commute with the Γ^*-action. We now draw page 2:

Example


bz
ba^2y
a^4w
a^2z
a^4y
a^2x
baw
bay
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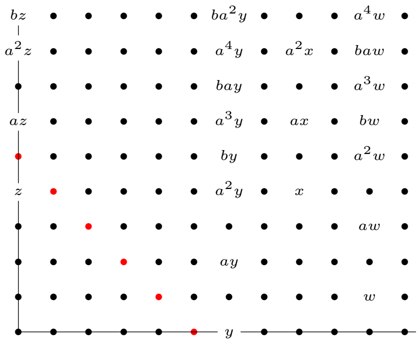
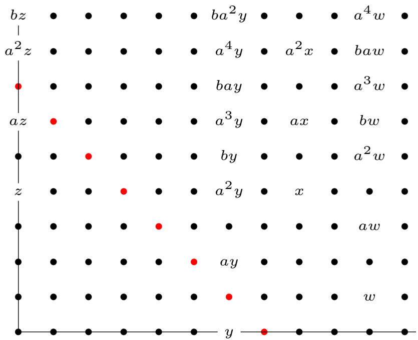
Example



The generator z survives to E_∞ since $\deg z = (0,4)$ and $d_r(z)$ must have total degree 5, but $E_2^{\bullet\bullet}$ is trivial there.

Example



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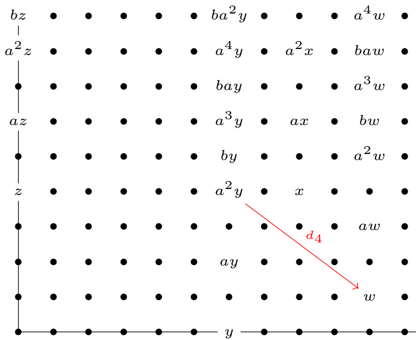

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This argument doesn't work for w, since $d_4(a^2y) = w$.

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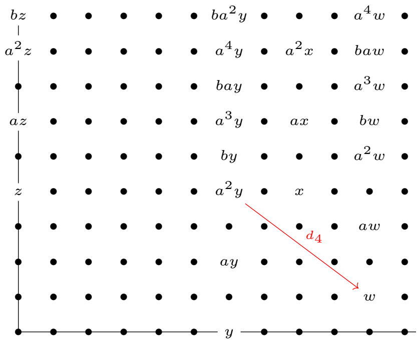

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This argument doesn't work for w, since $d_4(a^2y) = w$. But differentials commute with the action, so $d_4(a^2y) = a^2 d_4(y) = 0$. Thus, w survives, since it cannot be hit by any Γ^*-multiple of x or y.

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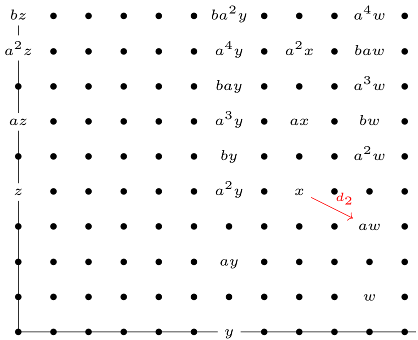

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Finally, $d_2(x) = aw$.

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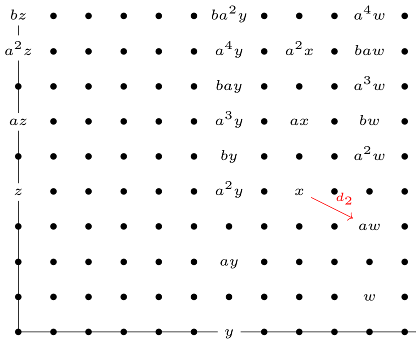

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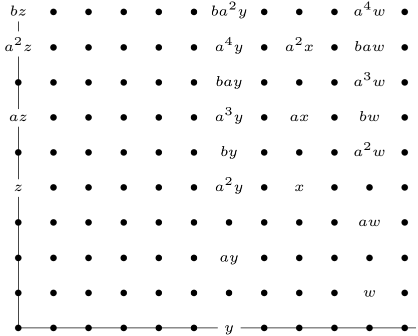

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This argument doesn't work for w, since $d_4(a^2y) = w$. But differentials commute with the action, so $d_4(a^2y) = a^2 d_4(y) = 0$. Thus, w survives, since it cannot be hit by any Γ^*-multiple of x or y.
Finally, $d_2(x) = aw$. We reach a contradiction by realizing $0 = d_2(0) = d_2(bx) = bd_2(x) = baw \neq 0$, so $d_2(x) = 0$. $d_r(x) = 0$ for $r > 2$ since arrows will go down and right and increase total degree by 1, missing any other nonzero terms. Hence x survives to E_∞ too.

Definition

- A graded algebra $(H^*, \cdot)$ is **graded commutative (skew-commutative)** if

$$x \cdot y = (-1)^{pq} y \cdot x$$

for $x \in H^p$ and $y \in H^q$.

Example

Let $k = \mathbf{Q}$ and $(A^*, \cdot)$, $(B^*, \cdot)$ be free graded skew-commutative algebras. We show that there are only two possibilities for A^* and B^*. If x_{2n} generates A^* and $\deg x_{2n} = 2n$, then observe

$$x_{2n}{}^k \cdot x_{2n}{}^\ell = (-1)^{k\cdot 2n\cdot \ell\cdot 2n} x_{2n}{}^\ell \cdot x_{2n}{}^k = x_{2n}{}^\ell \cdot x_{2n}{}^k.$$

Hence we have honest commutativity, and $A^* \cong \mathbf{Q}[x_{2n}]$, the *polynomial algebra* on one generator of dimension $2n$.
On the other hand, if x_{2n+1} generates B^* with $\deg x_{2n+1} = 2n+1$, then

$$x_{2n+1} \cdot x_{2n+1} = (-1)^{(2n+1)(2n+1)} x_{2n+1} \cdot x_{2n+1} = -x_{2n+1} \cdot x_{2n+1},$$

we have $(x_{2n+1})^2 = (x_{2n+1})^{\geq 2} = 0$. Call $B^* = \Lambda(x_{2n+1})$, the *exterior algebra* on one generator of dimension $2n + 1$.

Example

Suppose there is a spectral sequence of algebras $\{E_r^{\bullet\bullet}, d_r\}$ such that $E_2^{\bullet\bullet} \cong V^* \otimes W^*$ as bigraded algebras, where V^* and W^* are graded algebras, and $E_2^{\bullet\bullet} \Rightarrow H^*$ as a graded algebra.
Suppose further that $H^* \cong \mathbf{Q}$ (where $\mathbf{Q}$ as a graded algebra is $H^0 = \mathbf{Q}$, $H^{>0} = 0$).
We claim if $V^* \cong \mathbf{Q}[x_{2n}]$, then $W^* \cong \Lambda(x_{2n-1})$ (and vice versa, if $V^* \cong \Lambda(x_{2n+1})$, then $W^* \cong \mathbf{Q}[x_{2n}]$).

Example

Let $V^* \cong \mathbf{Q}[x_{2n}]$. We show $W^* \cong \Lambda(x_{2n+1})$.
Recall that $d_r(v \otimes w)$ satisfies the Leibniz rule; i.e.,

$$d_r(v \otimes w) = d_r(v) \otimes w + (-1)^{\deg v} v \otimes d_r(w).$$

Also note that $d_r|_{V^*} = 0$ and $d_r|_{W^*}$ has image in $V^* \otimes W^*$. If $d_r(1 \otimes w) = \sum v_j \otimes w_j$, then observe that via the Leibniz rule,

$$d_r(1 \otimes w^k) = k \left(\sum v_j \otimes (w_j w^{k-1}) \right).$$

Example

We will build a page of the spectral sequence using the fact that we know our spectral sequence converges to $\mathbf{Q}$ to show that $W^* \cong \Lambda(x_{2n+1})$.

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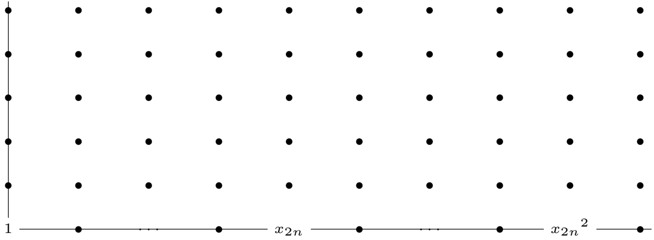

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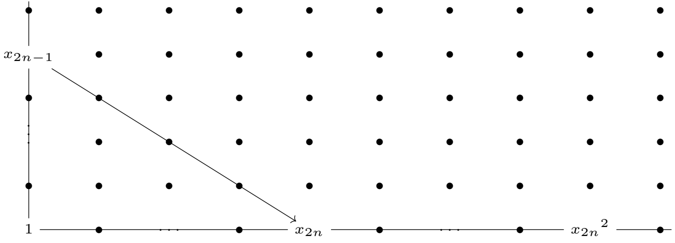

Since x_{2n} does not survive to E_∞, there exists $x_{2n-1} \in W^*$ such that $d_{2n}(1 \otimes x_{2n-1}) = x_{2n} \otimes 1$.

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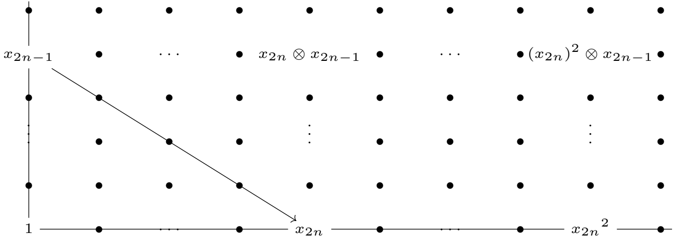

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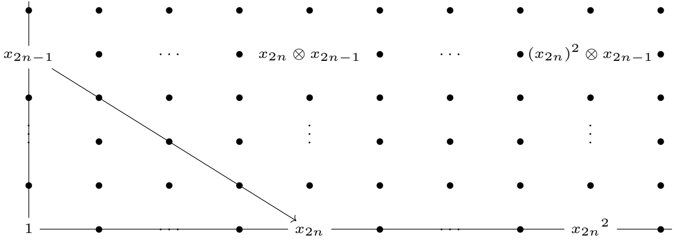

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The existence of x_{2n-1} generates new elements on page 2: $(x_{2n})^\ell \otimes x_{2n-1}$.
Since d_{2n-1} is a differential, hence derivation,

$$
\begin{aligned}
d_{2n-1}((x_{2n})^\ell \otimes x_{2n-1}) &= d_{2n-1}((x_{2n})^m) \otimes x_{2n-1} + (x_{2n})^m \otimes d_{2n-1}(x_{2n-1}) \\
&= md_{2n-1}(x_{2n})(x_{2n})^{m-1}) \otimes x_{2n-1} + (x_{2n})^{m+1} \otimes 1 \\
&= (x_{2n})^{m+1} \otimes 1.
\end{aligned}
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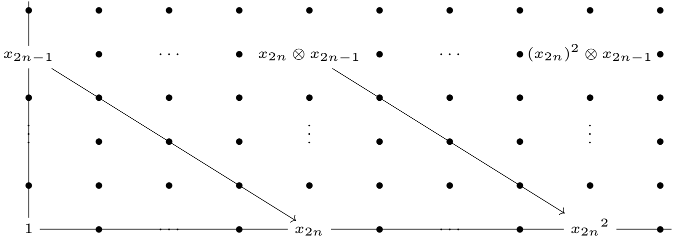

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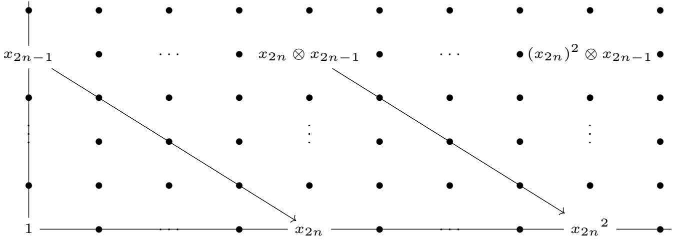
The existence of x_{2n-1} generates new elements on page 2: $(x_{2n})^\ell \otimes x_{2n-1}$.
Since d_{2n-1} is a differential, hence derivation,

$$\begin{aligned} d_{2n-1}((x_{2n})^\ell \otimes x_{2n-1}) &= d_{2n-1}((x_{2n})^m) \otimes x_{2n-1} + (x_{2n})^m \otimes d_{2n-1}(x_{2n-1}) \\ &= md_{2n-1}(x_{2n})(x_{2n})^{m-1}) \otimes x_{2n-1} + (x_{2n})^{m+1} \otimes 1 \\ &= (x_{2n})^{m+1} \otimes 1. \end{aligned}$$

Hence the arrows above.

Example

We will build a page of the spectral sequence using the fact that we know our spectral sequence converges to $\mathbf{Q}$ to show that $W^* \cong \Lambda(x_{2n+1})$.



Since x_{2n} does not survive to E_∞, there exists $x_{2n-1} \in W^*$ such that $d_{2n}(1 \otimes x_{2n-1}) = x_{2n} \otimes 1$.
The existence of x_{2n-1} generates new elements on page 2: $(x_{2n})^\ell \otimes x_{2n-1}$.
Since d_{2n-1} is a differential, hence derivation,

$$\begin{aligned} d_{2n-1}((x_{2n})^\ell \otimes x_{2n-1}) &= d_{2n-1}((x_{2n})^m) \otimes x_{2n-1} + (x_{2n})^m \otimes d_{2n-1}(x_{2n-1}) \\ &= m d_{2n-1}(x_{2n})(x_{2n})^{m-1}) \otimes x_{2n-1} + (x_{2n})^{m+1} \otimes 1 \\ &= (x_{2n})^{m+1} \otimes 1. \end{aligned}$$

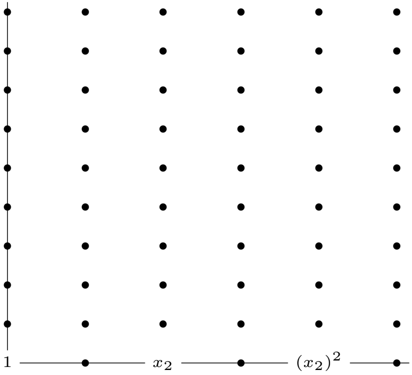
Hence the arrows above.
Finally, observe that if W^* had any other elements, they would give rise to classes that would persist to E_∞, contradicting that $H^* = \mathbf{Q}$. Hence $W^* \cong \Lambda(x_{2n-1})$, as desired.

Example

Suppose there is a spectral sequence of algebras $\{E_r^{\bullet\bullet}, d_r\}$ such that $E_2^{\bullet\bullet} \cong V^* \otimes W^*$ as bigraded algebras, where V^* and W^* are graded algebras, and $E_2^{\bullet\bullet} \Rightarrow H^*$ as a graded algebra.
Suppose further that $H^* \cong \mathbf{Q}$.
We claim if $V^* \cong \mathbf{Q}[x_2] / (x_2)^3$, then $W^* \cong \Lambda(x_1) \otimes \mathbf{Q}[z]$.

Example

Another proof by building page 2. We start with



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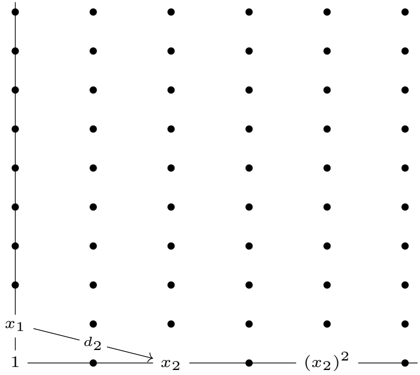

First, we need $\Lambda(x_1)$ in W^* such that $d_2(x_1) = x_2$.

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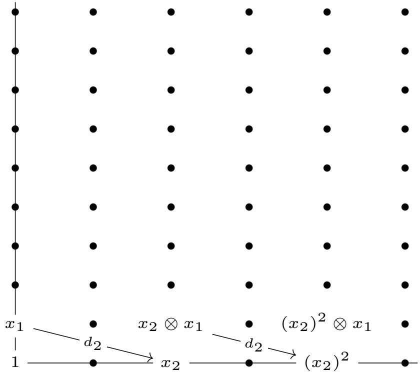

First, we need $\Lambda(x_1)$ in W^* such that $d_2(x_1) = x_2$.
We get the following elements and maps.

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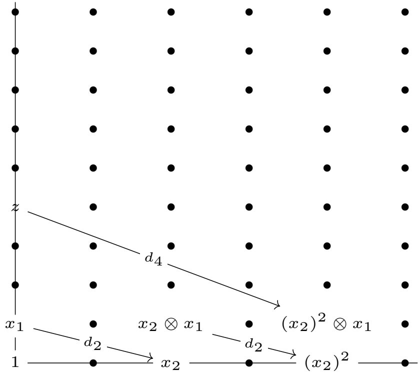

First, we need $\Lambda(x_1)$ in W^* such that $d_2(x_1) = x_2$.
We get the following elements and maps.
We need to take care of $(x_2)^2 \otimes x_1$. It cannot map to anything, so we need an element to map to it. Since $(x_2)^2 \otimes x_1$ has total degree 5, we need $z \in W^*$ with degree 4 such that $d_4(z) = (x_2)^2 \otimes x_1$.

Example

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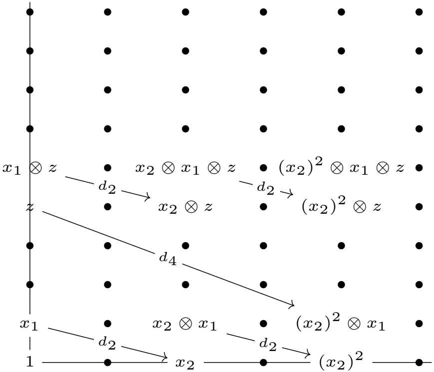

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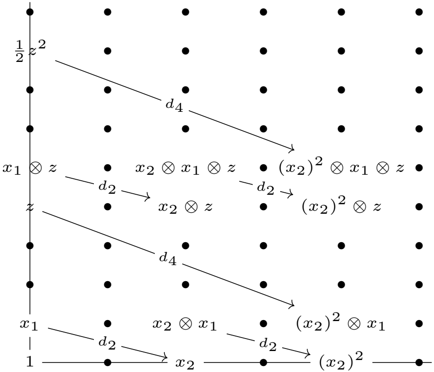

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We get the following elements and maps.
Observe via computation that $d_4\left(\frac{1}{2}z^2\right) = (x_2)^2 \otimes (x_1 \otimes z)$.

Example

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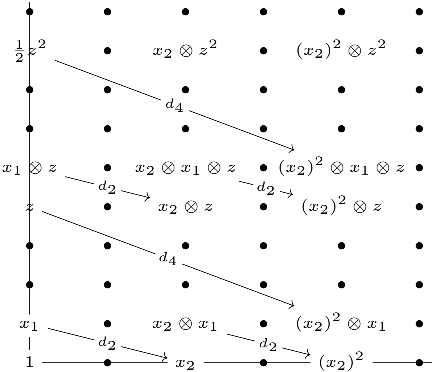

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We get the following elements and maps.
Observe via computation that $d_4\left(\frac{1}{2}z^2\right) = (x_2)^2 \otimes (x_1 \otimes z)$.
The pattern continues.